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Possible evidence of Kaluza–Klein particles in a scalar model with spherical compactification

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Abstract. Possible experimental manifestations of the contribution of heavy Kaluza–Klein particles, within a simple scalar model in six dimensions with spherical compactification, are studied. The approach is based on the assumption that the inverse radius L^{-1} of the space of extra dimensions is of the order of the scale of the supersymmetry breaking $M_{\mathrm{SUSY}} \sim 1 \div 10$ TeV. The total cross section of the scattering of two light particles is calculated to one-loop order and the effect of the Kaluza–Klein tower is shown to be noticeable for energies $\sqrt{s} \geqslant 1.4L^{-1}$.

1. Introduction

Many of the modern approaches extending the standard model include the hypothesis multidimensionality of spacetime (e.g. Kaluza–Klein-type theories, supergravity, superstring theory; see for instance [1] and [2], and references therein). The extra dimensions are supposed to be compactified, i.e. 'curled up' to a compact manifold of a characteristic scale L.

Various aspects of multidimensional models of gravity and particle interactions at the classical level have been intensively studied (see, for example, [3, 4] for reviews). Some issues related to quantum features of Kaluza–Klein theories were considered in [5–9]. One of the interesting problems is the search for and calculation of characteristic effects related to the multidimensional nature of Kaluza–Klein theories. In this paper, we consider one of these effects, which is essentially quantum.

As is well known, by performing mode expansion, a multidimensional model on the spacetime $M^4 \times K$ (where K is a compact manifold) can be represented as an effective theory on M^4 with an infinite set of particles, which is often referred to as the Kaluza–Klein tower of particles or modes. The spectrum of the four-dimensional theory depends on the topology and geometry of K. The sector of the lowest state (of the zero mode) describes light particles (in the sense that their masses do not depend on L^{-1}) and coincides with the dimensionally reduced theory. Higher modes correspond to heavy particles with masses $\sim L^{-1}$ called *pyrgons* (from the Greek $\pi\upsilon\rho\gamma o\varsigma$, for ladder). It is the contribution

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of these pyrgons to physical quantities that might give evidence about the existence of extra dimensions.

As was demonstrated in [10, 11], in spite of the infinite number of fields in the theory on M^4—and, related to this, non-renormalizability—its contributions of the heavy particles after subtraction of the ultraviolet divergences decouple if the energy of the process is sufficiently small that $sL \ll 1$. This ensures that the low-energy limit of a Kaluza–Klein model is simply the dimensionally reduced theory with zero modes only. When the energies are comparable to L^{-1}, the heavy-mode contributions are not negligible. One might expect that, due to the infinite number of states in the Kaluza–Klein tower, some accumulation heavy-mode contributions takes place, which would lead to a noticeable effect even for energies $\sqrt{s} < 2M_1$ (M_1 is the mass of the first heavy mode) when direct production of the excited states is still not possible. In fact, because of the decoupling, only a few of the lowest states (with masses up to about $10L^{-1}$) will contribute essentially at these energies.

The purpose of our paper is to demonstrate that a noticeable effect does indeed appear. With this aim, we will choose a simple scalar ϕ^4-model on the six-dimensional spacetime $M^4 \times S^2$. The space of extra dimensions is the two-dimensional sphere S^2 of radius L upon which there is an $SO(3)$-invariant metric. Study of quantum effects on spheres (mainly calculations of the Casimir effect and effective potentials) can be found, for example, in [6, 7]. In this paper, we will obtain the total cross section for the (2 light particles) $\longrightarrow$ (2 light particles) scattering process. Pyrgons do not contribute at the tree level in this model. Actually, the discussion given in section 2 shows that this statement is rather general and is valid, for example, for any scalar model. Thus, the one-loop correction is important for including the effect due to the presence of heavy Kaluza–Klein modes. We will calculate the one-loop contribution and analyse it for a wide range of scattering-particle energies. A similar problem for torus compactification (i.e. when the space of extra dimensions K is the two-dimensional torus T^2) was considered in [12]. We will use the results of that paper for comparison with the results obtained in the present paper with the aim of understanding the extent to which the cross section may depend on the topology of the space of extra dimensions. We believe that our results are generic, and that comparable effects could actually be obtained in more realistic theories constructed in the framework of the Kaluza–Klein approach. We would like to mention that some results on the calculation of one-loop Feynman diagrams on the spacetime $M^4 \times T^N$ can be found in [9].

A similar problem in the context of orbifold compactifications of the heterotic superstring has been considered in [13]. There, the authors calculated some physical proccesses and estimated the inverse size of extra dimensions L^{-1} to be of the order $0.1 \div 10$ TeV depending on the model. Some earlier computations of upper bounds on the size L of extra dimensions using results of high-energy experiments with the assumption that even the first heavy Kaluza–Klein mode is not observed experimentally can be found in [14]. In many models, the spontaneous compactification mechanism gives L of the order of the inverse Planck mass M_{Pl}^{-1} (see, for example, [15] and the reviews [4]). In this case additional dimensions could reveal themselves only as peculiar gravitational effects or at an early stage of the evolution of the universe.

It is true that no natural mechanism providing compactification of the space of extra dimensions with larger scale is known so far. However, there are several different motivations for considering this case. One of them comes from Kaluza–Klein cosmology and stems from the fact that the density of heavy Kaluza–Klein particles cannot be too large the critical density of the universe will be exceeded. Estimates obtained in [16] give the bound $L^{-1} < 10^6$ GeV. Another motivation is based on the results of [17, 18]. The point is that relating L to the supersymmetry breaking scale M_{SUSY} leads to the cancellation of unwanted threshold corrections in superstring theories for certain superstring models with orbifold compactifications [18]. This opens the possibility of having quite attractive models where perturbative calculations are reliable and the perturbative breaking of supersymmetry can be implemented. The latter implies the existence of Kaluza–Klein modes with masses at the TeV scale (see [19]). Estimates of the magnitude of L^{-1} in [13], cited above, were obtained in this setting.

As already mentioned, due to its multidimensional nature, the complete theory with an infinite number of modes is non-renormalizable. This is not a basic difficulty if the theory is considered as some kind of low-energy limit of a more fundamental theory. Then renormalization counterterms of the multidimensional model could, in principle, be calculated in the fundamental theory, and the corresponding coupling constants should be considered as *phenomenological* constants. Fortunately, the contributions of the counterterms of higher-dimensional operators to the finite parts of the amplitudes are of the order $(s/\Lambda^2)^n$ [2], where $\sqrt{s}$ is the energy of the colliding particles, Λ can be regarded as a characteristic scale of the fundamental theory ($\Lambda \sim M_{\rm Pl}$ in the case of superstrings) and $n \geqslant 1$. Thus, when $\sqrt{s} \leqslant L^{-1} \ll \Lambda$, these contributions can be neglected.

Now, if, for instance, the fundamental theory turns out to be one of the superstring models with $L^{-1} \sim M_{\rm SUSY}$, discussed above, then the same value of L^{-1} should be taken in the corresponding effective Kaluza–Klein-type theory. This explains why in our computations we consider the possibility of L^{-1} being of the order of a few TeV and this enables us not only to see the type of effect that the extra dimensions give rise to, but also to estimate its order of magnitude, which, we believe, does *not* depend on the precise form of the model. The conclusion is that it should be possible to observe the effect in future experiments at supercolliders. Similar calculations for more realistic models and processes will be carried out in a forthcoming publication.

This paper also contains a section of more mathematical character, in which some zeta-function regularization techniques [20] that provide a rather elegant way of treating sums over Kaluza–Klein modes which appear in the theory are described (for a review of these methods see [21]). Though there is some literature on the technique of performing calculations on spheres (see, for example, [6, 7, 9] and references therein), we think that this part of our work is interesting in its own right, since several of the formulae that appear here are new and provide a non-trivial alternative method for dealing with spherical compactification. Some general results on calculations on curved spacetime can be found in the review [22].

The paper is organized as follows. In section 2, we describe the model, choose the renormalization condition and discuss the general structure of the one-loop results. In section 3, some useful asymptotic expansions of the corresponding (regularizing) zeta function are derived. Methods for the numerical evaluation of the one-loop contribution are devised and the amplitude for a wide range of energies is calculated. Section 4 contains an analysis of the total cross section in the multidimensional model, and it is compared with the cross section for theories with a finite number of heavy particles, as well as with the cross section obtained for torus compactification. Concluding remarks are presented in section 5. The appendix contains relevant formulae for the Epstein–Hurwitz zeta function and explicit expressions of the scattering amplitude, which are used for numerical computations in the main text.

2. Description of the model, mode expansion and renormalization

Let us consider a one-component scalar field on the six-dimensional manifold $E = M^4 \times S^2$, where M^4 is Minkowski spacetime and S^2 is the two-dimensional sphere of radius L. In spite of its simplicity, this model captures many interesting features of both the classical and quantum properties of multidimensional theories. The action is given by

$$S = \int_E d^4x\, d\Omega \left[-\frac{1}{2}\left(\frac{\partial\phi(x,\theta)}{\partial x^\mu}\right)^2 - \frac{1}{2}g^{ij}\frac{\partial\phi(x,\theta)}{\partial\theta^i}\frac{\partial\phi(x,\theta)}{\partial\theta^j} - \frac{1}{2}m_0^2\phi^2(x,\theta) - \frac{\hat{\lambda}}{4!}\phi^4(x,\theta)\right] \tag{1}$$

where x^μ, $\mu = 0, 1, 2, 3$, are the coordinates on M^4, θ^1 and θ^2 are the standard angular coordinates on S^2, $0 < \theta^1 < \pi$, $0 < \theta^2 < 2\pi$, and $d\Omega$ is the integration measure on the sphere. g_{ij} is the standard $SO(3)$-invariant metric on the two-dimensional sphere:

$$ds^2 = g_{ij}\, d\theta^i\, d\theta^j = L^2[(d\theta^1)^2 + \sin^2\theta^1 (d\theta^2)^2].$$

To re-interpret this model in four-dimensional terms, we make the expansion of the field $\phi(x,\theta)$

$$\phi(x,\theta) = \sum_{lm} \phi_{lm}(x) Y_{lm}(\theta) \tag{2}$$

where $l = 0, 1, 2, \ldots$, $m = -l, -l+1, \ldots, l-1, l$, and $Y_{lm}(\theta)$ are the eigenfunctions of the Laplace operator on the internal space, i.e. spherical harmonics, satisfying

$$\Delta Y_{lm} = -\frac{l(l+1)}{L^2} Y_{lm} \tag{3}$$

$$\int d\Omega\, Y_{lm}^* Y_{l'm'} = \delta_{l,l}\delta_{m,m'}. \tag{4}$$

Substituting this expansion into the action and integrating over θ, one obtains

$$S = \int_{M^4} d^4x \left\{ -\frac{1}{2}\left(\frac{\partial\phi_0(x)}{\partial x^\mu}\right)^2 - \frac{1}{2}m_0^2\phi_0^2(x) - \frac{\lambda_1}{4!}\phi_0^4(x) - \sum_{l>0}\sum_m \left[\frac{\partial\phi_{lm}^*(x)}{\partial x^\mu}\frac{\partial\phi_{lm}(x)}{\partial x_\mu} + M_l^2\phi_{lm}^*(x)\phi_{lm}(x)\right] - \frac{\lambda_1}{2}\phi_0^2(x)\sum_{l>0}\sum_m \phi_{lm}^*(x)\phi_{lm}(x)\right\} - S'_{\text{int}} \tag{5}$$

where the four-dimensional coupling constant λ_1 is related to the multidimensional coupling constant $\hat{\lambda}$ by $\lambda_1 = \hat{\lambda}/(\text{volume } S^2)$. In equation (5), S'_{int} includes all terms containing third and fourth powers of ϕ_{lm} with $l > 0$. We see that the model includes one real scalar field $\phi_0 \equiv \phi_{00}(x)$ describing a light particle of mass m_0 and an infinite set ('tower') of massive complex fields $\phi_{lm}(x)$ corresponding to heavy particles, or pyrgons, with masses given by

$$M_l^2 = m_0^2 + l(l+1)M \tag{6}$$

where $M = L^{-1}$ is the compactification scale.

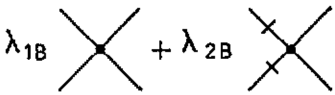

Figure 1. Tree diagrams contributing to the four-point Green function $\Gamma^{(\infty)}$. The lines correspond to the light particle, the bar corresponds to derivatives with respect to the external momenta.

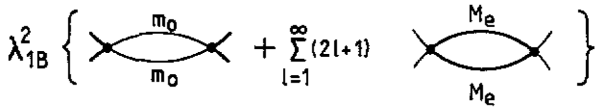

Figure 2. One-loop diagrams contributing to the four-point Green function $\Gamma^{(\infty)}$. Thin curves correspond to the light particle with mass m_0. The thick curve labelled M_l corresponds to propagation of the particle with mass M_l.

Let us consider the four-point Green function $\Gamma^{(\infty)}$ with external legs corresponding to the light particles ϕ_0. The index '∞' will indicate that the whole Kaluza–Klein tower of modes is taken into account. Diagrams which contribute to this function in the tree and one-loop approximations are presented in figures 1 and 2. Terms included in S'_{int} in equation (5) are not relevant to our computations.

Let us first analyse the tree-level contribution. The first diagram in figure 1 is exactly the same as in the case of the dimensionally reduced theory with the action given by the first line in equation (5). The second diagram appears due to extra divergences in the theory and is discussed below. Heavy modes do not contribute at this level. Such a property is rather general and is valid for all processes of the type (n light particles) $\longrightarrow$ (k light particles)—at least in all scalar multidimensional theories with polynomial interactions. The reason for this is simple. Suppose that we have a theory on $M^4 \times K$, where K is a compact space, with a polynomial interaction. The analysis of tree graphs shows that, in order to obtain a contribution of heavy modes at the tree level, one needs to have at least one vertex at which q light modes $\phi_0(x)$ (with $q < n + k$ for $(n + k) > 2$) interact with one heavy mode $\phi_N(x)$, where the generalized index N corresponds to a non-zero eigenvalue of the Laplace operator on K. After substituting the expansion of the multidimensional field over the eigenfunctions of this operator (analogous to (2)) into the original multidimensional action, the interaction term

$$\int_{M^4} \mathrm{d}^4x\, \phi_N(x)(\phi_0(x))^q$$

corresponding to this vertex, will always appear multiplied by the factor

$$\int_K \mathrm{d}\Omega\, Y_N(\theta)\, \underbrace{Y_0 \ldots Y_0}_{q \text{ times}}.$$

However, since the eigenfunction Y_0 corresponding to the zero eigenvalue of the Laplace operator on the compact manifold is a constant†, the above integral will always be zero,

† The authors thank C Nash for useful discussions on this issue.

due to the orthonormality condition. This implies, for example, that the decay process of a heavy mode with number N into q zero modes is forbidden for the class of models under consideration. In the case $K = S^2$, the feature discussed here is a manifestation of the conservation of angular momentum.

Let us now analyse the one-loop correction in our model. It is easy to check that, owing to the infinite sum of diagrams (see figure 2), the Green function to one-loop order is quadratically divergent. This is certainly a reflection of the fact that the original theory is, in fact, six-dimensional and, therefore, non-renormalizable. Thus, the divergences cannot be removed by renormalization of the coupling constant in equation (5) alone. We must also add a counterterm $\lambda_{2B}\phi^2(x,y)\Box_{(4+d)}\phi^2(x,y)$, where $\Box_{(4+d)}$ is the D'Alembertian on E and λ_{2B} has mass-dimension two. The second diagram in figure 1 corresponds to the contribution of this counterterm. Of course, for the calculation of the other Green functions, or of higher-order loop corrections, other types of counterterms are necessary, but we are not going to discuss them here. Hence, the Lagrangian we will use for our investigation is

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial\phi_0(x)}{\partial x^\mu}\right)^2 - \frac{m_0^2}{2}\phi_0^2(x) - \sum_{l>0}\sum_m\left[\frac{\partial\phi_{lm}^*(x)}{\partial x^\mu}\frac{\partial\phi_{lm}(x)}{\partial x_\mu} + M_l^2\phi_{lm}^*(x)\phi_{lm}(x)\right]$$
$$- \frac{\lambda_{1B}}{4!}\phi_0^4(x) - \frac{\lambda_{1B}}{2}\phi_0^2(x)\sum_{l>0}\sum_m\phi_{lm}^*(x)\phi_{lm}(x) - \frac{\lambda_{2B}}{4!}\phi_0^2(x)\Box\phi_0^2(x) \tag{7}$$

where λ_{1B} and λ_{2B} are bare coupling constants. To regularize the four-dimensional integrals, corresponding to the one-loop diagrams of figure 2, we will employ dimensional regularization, which is performed, as usual, by performing an analytical continuation of the integrals to $(4-2\epsilon)$ dimensions. κ will be a mass scale set up by the regularization procedure. The sums over l will be regularized by means of the zeta-function technique [20, 21].

Let us now specify the renormalization scheme. It is known that the results of the calculation of physical quantities at a finite order of perturbation theory depend on the renormalization scheme. Since our goal is basically to study the difference between the contributions corresponding to the complete Kaluza–Klein tower of particles and to the light particle only, we should choose a scheme in which the end results are minimally affected by the renormalization procedure. A reasonable method is to impose the condition that the physical amplitude for the (2 light particles) $\longrightarrow$ (2 light particles) scattering process of the complete theory and the amplitude of the same process in the four-dimensional theory with the zero mode only (i.e. the theory given by Lagrangian (7) with all non-zero modes and the last term omitted) must coincide at some normalization (subtraction) point corresponding to low energies. As a subtraction point, we will choose the following point in the space of invariant variables built up out of the external four-momenta p_i $(i = 1, 2, 3, 4)$ of the scattering particles:

$$\begin{gathered} p_1^2 = p_2^2 = p_3^2 = p_4^2 = m_0^2 \\ p_{12}^2 = s = \mu_s^2 \qquad p_{13}^2 = t = \mu_t^2 \qquad p_{14}^2 = u = \mu_u^2 \end{gathered} \tag{8}$$

where $p_{1j}^2 = (p_1 + p_j)^2$, $j = 2, 3, 4$, and s, t and u are the Mandelstam variables. Since the subtraction point is located on the mass shell, it satisfies the standard relation $\mu_s^2 + \mu_t^2 + \mu_u^2 = 4m_0^2$. The renormalization prescription formulated above can be written as

$$\Gamma^{(\infty)}(p_{1j}^2; m_0, M, \lambda_{1B}, \lambda_{2B}, \epsilon)|_{\text{s.p.}} = \Gamma^{(0)}(p_{1j}^2; m_0, \lambda_{1B}', \epsilon)|_{\text{s.p.}} = g\kappa^{2\epsilon} \tag{9}$$

$$\left[\frac{\partial}{\partial p_{12}^2} + \frac{\partial}{\partial p_{13}^2} + \frac{\partial}{\partial p_{14}^2}\right]\Gamma^{(\infty)}\Bigg|_{\text{s.p.}} = \left[\frac{\partial}{\partial p_{12}^2} + \frac{\partial}{\partial p_{13}^2} + \frac{\partial}{\partial p_{14}^2}\right]\Gamma^{(0)}\Bigg|_{\text{s.p.}} + \frac{\lambda_2}{4}\kappa^{-2+2\epsilon}. \tag{10}$$

Here, $\Gamma^{(0)}$ is the four-point Green function of the four-dimensional theory with zero-mode field only (i.e. the dimensionally reduced theory), λ'_{1B} being its bare coupling constant. In equation (9), we have written down explicitly the dependence of the Green functions on the momentum arguments and parameters of the theory, and we have taken into account that to one-loop order they depend on p_{12}^2, p_{13}^2 and p_{14}^2 only. The label s.p. means that the corresponding quantities are taken at the subtraction point (8). g and λ_2 are renormalized coupling constants; the latter are included for the sake of generality only, and we will see that our final result does not depend on it.

For the usual $\lambda\phi_0^4$-theory in four dimensions, the renormalization condition given by (9) alone is sufficient, whereas both conditions (9) and (10) are necessary for subtracting the ultraviolet divergences of theory (7), because of the presence of additional divergences due to its multidimensional character.

To one-loop order, the Green functions of the complete theory and of the theory with the zero mode only, are given, respectively, by

$$\begin{aligned}\Gamma^{(\infty)}(p_{1j}^2; m_0, M, \lambda_{1B}, \lambda_{2B}, \epsilon) &= \lambda_{1B} + \lambda_{2B}\frac{p_{12}^2 + p_{13}^2 + p_{14}^2}{12} \\ &\quad + \lambda_{1B}^2[K_0(p_{1j}^2; m_0, \epsilon) + \Delta K(p_{1j}^2; m_0, M, \epsilon)] \\ \Gamma^{(0)}(p_{1j}^2; m_0, \lambda'_{1B}, \epsilon) &= \lambda'_{1B} + \lambda'^2_{1B} K_0(p_{1j}^2; m_0, \epsilon).\end{aligned} \tag{11}$$

Here

$$K_0(p_{1j}^2; m_0, \epsilon) \equiv K_{00}(p_{1j}^2; m_0^2, \epsilon) \qquad \Delta K(p_{1j}^2; m_0, M, \epsilon) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} K_{lm}(p_{1j}^2; M_l^2, \epsilon) \tag{12}$$

and K_{lm} is the contribution of mode ϕ_{lm} with mass $M_l = \sqrt{m_0^2 + M^2 l(l+1)}$ (see equation (6)) to the one-loop diagram of two light particles scattering

$$K_{lm}(p_{1j}^2; M_l^2, \epsilon) = \frac{-\mathrm{i}}{32\pi^4 M_l^{2\epsilon}}\left[I\left(\frac{p_{12}^2}{M_l^2}, \epsilon\right) + I\left(\frac{p_{13}^2}{M_l^2}, \epsilon\right) + I\left(\frac{p_{14}^2}{M_l^2}, \epsilon\right)\right]. \tag{13}$$

K_0 corresponds to the first diagram in figure 1 and ΔK corresponds to the contribution of the sum term in figure 2. Here we assume that $\lambda_{2B} \sim \lambda_{1B}^2$, so that the one-loop diagrams proportional to $\lambda_{1B}\lambda_{2B}$ or λ_{2B}^2 can be neglected. It can be proven that this hypothesis is consistent (see [10]). Note that if we did not make this assumption, proper control of the proliferation of the divergences would become very difficult. Function I, in the formula above, is the standard one-loop integral

$$\begin{aligned}I\left(\frac{p^2}{M^2}, \epsilon\right) &= M^{2\epsilon}\int \mathrm{d}^{4-2\epsilon}q\, \frac{1}{(q^2+M^2)((q-p)^2+M^2)} \\ &= \mathrm{i}\pi^{2-\epsilon}\Gamma(\epsilon)M^{2\epsilon}\int_0^1 \mathrm{d}x\, \frac{1}{[M^2 - p^2x(1-x)]^\epsilon}.\end{aligned} \tag{14}$$

Let us also introduce the sum of the one-loop integrals over all the Kaluza–Klein modes

$$\Delta I\left(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon\right) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(\frac{M^2}{M_l^2}\right)^\epsilon I\left(\frac{p^2}{M_l^2}, \epsilon\right) \tag{15}$$

so that

$$\Delta K(p_{1j}^2; m_0, M, \epsilon)$$
$$= \frac{-\mathrm{i}}{32\pi^4 M^{2\epsilon}} \left[\Delta I \left(\frac{p_{12}^2}{M^2}, \frac{m_0}{M}, \epsilon \right) + \Delta I \left(\frac{p_{13}^2}{M^2}, \frac{m_0}{M}, \epsilon \right) + \Delta I \left(\frac{p_{14}^2}{M^2}, \frac{m_0}{M}, \epsilon \right) \right].$$

Performing the renormalization, we obtain the following expression for the renormalized four-point Green function:

$$\Gamma_R^{(\infty)} \left(\frac{p_{1j}^2}{\mu_j^2}; \frac{\mu_j^2}{M^2}; \frac{m_0}{M}, \frac{\kappa}{M}, g, \lambda_2 \right) = \lim_{\epsilon \to 0} \kappa^{-2\epsilon} \Gamma^{(\infty)}(p_{1j}^2; m_0, M, \lambda_{1\mathrm{B}}(g, \lambda_2), \lambda_{2\mathrm{B}}(g, \lambda_2), \epsilon)$$
$$= \lim_{\epsilon \to 0} \Bigg\{ g + \lambda_2 \frac{p_{12}^2 + p_{13}^2 + p_{14}^2 - \mu_s^2 - \mu_t^2 - \mu_u^2}{12\kappa^2}$$
$$+ g^2 \kappa^{2\epsilon} \Bigg[K_0(p_{1j}^2; m_0, \epsilon) - K_0(\mu_j^2; m_0, \epsilon)$$
$$+ \Delta K(p_{1j}^2; m_0, M, \epsilon) - \Delta K(\mu_j^2; m_0, M, \epsilon)$$
$$- \frac{p_{12}^2 + p_{13}^2 + p_{14}^2 - \mu_s^2 - \mu_t^2 - \mu_u^2}{3} \left(\frac{\partial}{\partial p_{12}^2} + \frac{\partial}{\partial p_{13}^2} + \frac{\partial}{\partial p_{14}^2} \right)$$
$$\times \Delta K(p_{1j}^2; m_0, M, \epsilon)|_{\mathrm{sp}} \Bigg] \Bigg\} \tag{16}$$

where we denote $\mu_2^2 = \mu_s^2$, $\mu_3^2 = \mu_t^2$ and $\mu_4^2 = \mu_u^2$. The right-hand side of this expression is regular in ϵ and, after calculating the integrals and the sums over l and m, we take the limit $\epsilon \to 0$. The above expression is rather general and valid for any arbitrary subtraction point. It simplifies if the relation $\mu_s^2 + \mu_t^2 + \mu_u^2 = 4m_0^2$, fulfilled by subtraction point (8), is taken into account. Before doing this, let us discuss the structure of the renormalized Green function for a subtraction point where all subtraction scales are equal: $\mu_s^2 = \mu_t^2 = \mu_u^2 = \mu^2$. Then equation (16) can be written as

$$\Gamma_R^{(\infty)} \left(\frac{p_{1j}^2}{\mu^2}; \frac{\mu^2}{M^2}; \frac{m_0}{M}, \frac{\kappa}{M}, g, \lambda_2 \right) = g + \lambda_2 \frac{p_{12}^2 + p_{13}^2 + p_{14}^2 - 3\mu^2}{12\kappa^2}$$
$$+ g^2 \lim_{\epsilon \to 0} \kappa^{2\epsilon} \Bigg\{ K_0(p_{1j}^2; m_0, \epsilon) - K_0(\mu^2, \mu^2, \mu^2; m_0, \epsilon)$$
$$- \frac{\mathrm{i}}{32\pi^4} \frac{1}{M^{2\epsilon}} \Bigg[\Bigg(\Delta I \left(\frac{p_{12}^2}{M^2}, \frac{m_0}{M}, \epsilon \right) - \Delta I \left(\frac{\mu^2}{M^2}, \frac{m_0}{M}, \epsilon \right)$$
$$- (p_{12}^2 - \mu^2) \frac{\partial}{\partial p_{12}^2} \Delta I \left(\frac{p_{12}^2}{M^2}, \frac{m_0}{M}, \epsilon \right) \Bigg|_{p_{12}^2 = \mu^2} \Bigg)$$
$$+ (p_{12}^2 \longrightarrow p_{13}^2) + (p_{12}^2 \longrightarrow p_{14}^2) \Bigg] \Bigg\}. \tag{17}$$

As mentioned above, the contribution ΔI of the heavy modes contains quadratic divergences in momenta (in the framework of the dimensional regularization used here this means that it contains singular terms $\sim 1/\epsilon$ and $\sim p^2/\epsilon$). The renormalization prescription amounts to subtraction of the first two terms of the Taylor expansion of this contribution at the point μ^2, which is sufficient to remove the divergences. Contribution K_0 of the zero-mode sector (the dimensionally reduced theory) diverges only logarithmically and subtraction of the first term of the Taylor expansion—imposed by renormalization prescription (9)—is sufficient to make it finite in the limit $\epsilon \to 0$.

Finally, let us write down the expression for the four-point Green function of the complete theory (i.e. with all the Kaluza–Klein modes) renormalized according to conditions (9) and (10) at subtraction point (8) and taken at a momentum point which lies on the mass shell of the light particle. We get

$$\Gamma_R^{(\infty)}\left(\frac{s}{\mu_s^2}, \frac{t}{\mu_t^2}, \frac{u}{\mu_u^2}; \frac{\mu_s^2}{M^2}, \frac{\mu_t^2}{M^2}, \frac{\mu_u^2}{M^2}; \frac{m_0}{M}, g\right)$$
$$= g + g^2 \lim_{\epsilon\to 0} \kappa^{2\epsilon}\left[K_0(s,t,u;m_0,\epsilon) - K_0(\mu_s^2,\mu_t^2,\mu_u^2;m_0,\epsilon)\right.$$
$$\left.\times\ \Delta K(s,t,u;m_0,M,\epsilon) - \Delta K(\mu_s^2,\mu_t^2,\mu_u^2;m_0,M,\epsilon)\right]. \tag{18}$$

The variables s, t and u are not independent, since they satisfy the well known Mandelstam relation $s + t + u = 4m_0^2$.

The formula above is rather remarkable. It turns out that on the mass shell, due to cancellations among the s-, t- and u-channels, the contribution proportional to λ_2 and the terms containing derivatives of the one-loop integrals vanish. Thus, heavy Kaluza–Klein modes contribute to the renormalized Green function on the mass shell in exactly the same way as the light particle in the dimensionally reduced theory. Indeed, it can easily be checked that the additional non-renormalized divergences arising from the infinite summation in ΔK cancel among themselves when the three scattering channels are added together.

3. Calculation of the one-loop contribution

In this section, we will analyse the one-loop contribution of the heavy Kaluza–Klein modes, develop methods for its numerical evaluation and present results for the amplitude of (2 light particles) $\longrightarrow$ (2 light particles) scattering.

The starting point of the analysis is the expression

$$\Delta I\left(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon\right) = \mathrm{i}\pi^{2-\epsilon}\Gamma(\epsilon)\int_0^1 \mathrm{d}x \sum_{l,m}{}' \left(\frac{M^2}{M_l^2}\right)^{\epsilon}\left[1 - \frac{p^2 x(1-x)}{M_l^2}\right]^{-\epsilon} \tag{19}$$

(see (14) and (15)), where the prime means that the term for $l = 0$ is absent from the summatory. Performing the ϵ-expansion and integrating over the x-variable in equation (14), we obtain:

$$I\left(\frac{p^2}{M^2}, \epsilon\right) = \mathrm{i}\pi^{2-\epsilon}\Gamma(\epsilon) - \mathrm{i}\pi^2 J\left(\frac{p^2}{4M^2}\right) + \mathcal{O}(\epsilon) \tag{20}$$

where $J(z)$ is the finite part of the one-loop integral [23]

$$J(z) = \int_0^1 \mathrm{d}x \ln[1 - 4zx(1-x)] \tag{21}$$

which is equal to

$$J(z) = \begin{cases} J_1(z) = 2\sqrt{\dfrac{z-1}{z}}\ln\left(\sqrt{1-z}+\sqrt{-z}\right) - 2 & \text{for } z \leqslant 0 \\ J_2(z) = 2\sqrt{\dfrac{1-z}{z}}\tan^{-1}\sqrt{\dfrac{z}{1-z}} - 2 & \text{for } 0 < z \leqslant 1 \\ J_3(z) = -\mathrm{i}\pi\sqrt{\dfrac{z-1}{z}} + 2\sqrt{\dfrac{z-1}{z}}\ln\left(\sqrt{z}+\sqrt{z-1}\right) - 2 & \text{for } z > 1. \end{cases} \tag{22}$$

Now we can proceed with the summation over l and m. Taking the degeneracy in (19) into account and using ζ-regularization for the sums, we get

$$\Delta I\left(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon\right) = \mathrm{i}\pi^{2-\epsilon}\Gamma(\epsilon)[2\zeta(-1, \tfrac{1}{2}) - 1] - 2\mathrm{i}\pi^2\sum_{l=1}^{\infty}(l+\tfrac{1}{2})\ln[(l+\tfrac{1}{2})^2 + b]$$
$$- 2\mathrm{i}\pi^2\Delta J\left(\frac{p^2}{4M^2}; \frac{m_0}{M}\right) + \mathcal{O}(\epsilon). \tag{23}$$

In particular, for $p^2 < 0$,

$$\Delta J\left(\frac{p^2}{4M^2}; \frac{m_0}{M}\right) = \sum_{l=1}^{\infty}(l+\tfrac{1}{2})J_1\left(\frac{p^2}{4M_l^2}\right) \tag{24}$$

while for $p^2 > 0$

$$\Delta J\left(\frac{p^2}{4M^2}; \frac{m_0}{M}\right) = \sum_{l=1}^{l^*(p)}(l+\tfrac{1}{2})J_3\left(\frac{p^2}{4M_l^2}\right) + \sum_{l=l^*(p)+1}^{\infty}(l+\tfrac{1}{2})J_2\left(\frac{p^2}{4M_l^2}\right) \tag{25}$$

which contains, in general, an imaginary part. Here $l^*(p)$ is the maximum value of l which satisfies the inequality $4M^2(l+\frac{1}{2})^2 < p^2 - 4m_0^2$. If such an l does not exist, or if it is smaller than 1, we set $l^*(p) = 0$ and the first sum in equation (25) is absent. As already mentioned, the divergent sums over l are understood as being regularized by the zeta-function procedure. These formally divergent series, which are independent of p^2 or are linear in p^2, do not contribute to physical (renormalized) quantities (see the discussion in section 2).

The calculation is now performed by using the zeta-function method. As we clearly see from equation (23) above, the number of terms contributing to each sum changes with p. Thus, different explicit series are obtained for the different ranges of M^2/p^2. The first range $|p^2/4M_1^2| < 1$ is somewhat special and deserves careful treatement. Expanding the functions under the summation signs in powers of $u_l = p^2/(4M_l^2)$ we get

$$\Delta I\left(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon\right) = \mathrm{i}\pi^{2-\epsilon}\Gamma(\epsilon)[2\zeta(-1, \tfrac{1}{2}) - 1] - 2\mathrm{i}\pi^2\sum_{l=1}^{\infty}(l+\tfrac{1}{2})\ln[(l+\tfrac{1}{2})^2 + b]$$
$$+ 4\mathrm{i}\pi^2\sum_{l=1}^{\infty}(l+\tfrac{1}{2})\left(\frac{u_l}{3} + \frac{2u_l^2}{15} + \frac{8u_l^3}{105} + \frac{16u_l^4}{315} + \frac{128u_l^5}{3465} + \frac{256u_l^6}{9009} + \cdots\right)$$
$$+ \mathcal{O}(\epsilon)$$

$$u_l \equiv \frac{p^2}{4M_l^2} \equiv \frac{p^2/4M^2}{(l+1/2)^2 + b} \qquad b \equiv \frac{m_0^2}{M^2} - \frac{1}{4} \qquad |p^2| < 4m_0^2 + 8M^2 \tag{26}$$

which is valid for small values of m_0^2/M^2, including the case $m_0^2 = 0$. The sums over the eigenvalues for the spherical compactification give rise to the inhomogeneous Epstein–Hurwitz zeta functions

$$F(s; a, b) \equiv \sum_{l=1}^{\infty} [(l + a)^2 + b]^{1-s}. \tag{27}$$

We re-expand equation (26) to give an expression which is absolutely convergent in $p^2/(4M_1^2)$ whenever $m_0^2/M^2 < 5/2$ [6, 22, 24, 25]. In this way, we obtain

$$\begin{aligned} \Delta I\left(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon\right) = 2\mathrm{i}\pi^2 \Bigg[& -\frac{11}{24}\pi^{-\epsilon}\Gamma(\epsilon) + 2\zeta'(-1, 1/2) + \ln 2 \\ & + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \zeta^{(1)}(2k-1, 1/2) \left(\frac{m_0^2}{M^2} - \frac{1}{4}\right)^k \\ & + \sum_{k=1}^{\infty} c_k \left(\frac{m_0^2}{2M^2} + 1\right)^k h^{(1)}\left(k; \frac{1}{2}, \frac{m_0^2}{M^2} - \frac{1}{4}\right) \left(\frac{p^2}{4M_1^2}\right)^k + O(\epsilon^2) \Bigg]. \end{aligned} \tag{28}$$

Here

$$h^{(1)}(k; 1/2, b) \equiv \left(\frac{p^2}{4M^2}\right)^{-k} \sum_{l=1}^{\infty} (l + \tfrac{1}{2}) u_l^k. \tag{29}$$

Superscripts (1) mean 'truncated', in the sense that the first term in the definitions of these zeta functions (the one for $l = 0$) is absent. The following relation holds:

$$h^{(1)}(k; 1/2, b) = \frac{1}{1-k} \frac{\partial}{\partial a} F(k; a, b)|_{a=1/2} \tag{30}$$

where function F was introduced in equation (27). Some properties of these functions and numerical values for some of the coefficients c_k in equation (28) are given in the appendix. The term with $k = 1$, which is formally divergent (see equation (30)), disappears from the final expression after renormalization. Functions $\zeta(s, a)$ and $\zeta^{(1)}(s, a)$ are also defined in the appendix.

For higher positive values of p^2, contributions of a new type appear (corresponding to the first sum in equation (25)). The expression for $\Delta I(p^2/M^2, m_0/M, \epsilon)$, which converges for $4(m_0^2 + 2M^2) \leqslant p^2 < 4(m_0^2 + 6M^2)$, is given by equation (45) in the appendix. Equations (28), (45) and similar expansions for other intervals for higher momentum turn out to be quite effective for very accurate numerical computations of the total cross section of the (2 light particles) $\longrightarrow$ (2 light particles) scattering. Results are presented in the next section.

4. Calculation of the total cross section

In this section, we calculate the total cross section $\sigma^{(\infty)}(s)$ of the scattering process (2 light particles) $\longrightarrow$ (2 light particles) in the case when the whole Kaluza–Klein tower of heavy particles contributes, and we compare it with $\sigma^{(N)}(s)$, the cross section obtained for the case when only the first N modes are taken into account (i.e. modes with $l = 0, 1, 2, \ldots, N$). With this notation, $\sigma^{(0)}(s)$ is the cross section in the dimensionally reduced model, i.e. when only the light particle contributes. Such a comparison will be quite illuminating for understanding the relative contributions of the various heavy modes.

We have found that the quantity which describes the net effect due to the tower of heavy particles is the following ratio, which is built up from the total cross sections:

$$\Delta^{(\infty,0)}\left(\frac{s}{4M^2}; \frac{\mu_s^2}{M^2}, \frac{\mu_u^2}{M^2}, \frac{\mu_t^2}{M^2}; \frac{m_0}{M}\right) \equiv 16\pi^2 \frac{\sigma^{(\infty)}(s) - \sigma^{(0)}(s)}{g\sigma^{(0)}(s)}. \tag{31}$$

Using expression (18) for the four-point Green function, renormalized according to (9) and (10), we calculate the corresponding total cross sections and obtain that, to leading (one-loop) order in the coupling constant g, function (31) is equal to

$$\begin{aligned}\Delta^{(\infty,0)}&\left(\frac{s}{4M^2}; \frac{\mu_s^2}{M^2}, \frac{\mu_u^2}{M^2}, \frac{\mu_t^2}{M^2}; \frac{m_0}{M}\right)\\ &= -2\left\{\operatorname{Re} \Delta J\left(\frac{s}{4M^2}, \frac{m_0}{M}\right) + \frac{2}{s-4m_0^2}\int_{-(s-4m_0^2)}^{0} \mathrm{d}u\, \Delta J\left(\frac{u}{4M^2}; \frac{m_0}{M}\right)\right.\\ &\left.\quad - \Delta J\left(\frac{\mu_s^2}{4M^2}; \frac{m_0}{M}\right) - \Delta J\left(\frac{\mu_t^2}{4M^2}; \frac{m_0}{M}\right) - \Delta J\left(\frac{\mu_u^2}{4M^2}; \frac{m_0}{M}\right)\right\}. \end{aligned}\tag{32}$$

Here, we assume that $\mu_s^2, \mu_t^2, \mu_u^2 < 4m_0^2$.

For the numerical evaluation, we take the zero-mode particle to be much lighter than the first heavy mode and choose the subtraction point to be at the low-energy interval. Recall that $\mu_s^2 + \mu_t^2 + \mu_u^2 = 4m_0^2$. We take

$$\frac{m_0^2}{M^2} = 10^{-4} \qquad \frac{\mu_s^2}{m_0^2} = 10^{-2} \qquad \mu_u^2 = \mu_t^2 = \tfrac{1}{2}(4m_0^2 - \mu_s^2). \tag{33}$$

First of all, we observe that if the parameters of our theory satisfy $\mu_s^2 \ll m_0^2 \ll M^2$ (notice that for our choice (33) these inequalities are indeed fulfilled), the dependence of the function $\Delta^{(\infty,0)}$ on μ_s^2/M^2, μ_t^2/M^2, μ_u^2/M^2 and m_0^2/M^2 is very weak, and in practice it depends only on one dimensionless parameter. We choose this parameter to be $z = s/(4M_1^2)$.

Using the results of section 3 we can calculate the function $\Delta^{(\infty,0)}(z)$. Its plot in the range $0 < z < 1$ is presented in figure 3. We see that the contribution of the Kaluza–Klein tower of particles is considerable. Thus, $\Delta^{(\infty,0)} \simeq 0.51$ for $s = 0.5(4M_1^2)$ and $\Delta^{(\infty,0)} \simeq 0.12$ for $s = 0.25(4M_1^2)$. Therefore, for energies much smaller than the one corresponding to the threshold of the first heavy particle, the effect is already quite noticeable.

We also note the quick convergence of the sums over l in equations (24), (25) and (28), contributing to the function $\Delta^{(\infty,0)}(z)$. A few terms already give curves which do not change when more summands are added (we have checked indeed that the sum up to $l = 50$ in the above series is, for all practical purposes, identical to the sum of the 20 first terms). Owing to the convergence of the sums, only the first few terms with low l give essential contributions, whereas those corresponding to the higher modes are negligible. To understand how many modes we really see from the plot of $\Delta^{(\infty,0)}$, further analysis is needed. With this purpose, in figure 3 we also present the curve $\Delta^{(1,0)}(z)$, characterizing the contribution of the first heavy mode only. We observe that the difference is not small

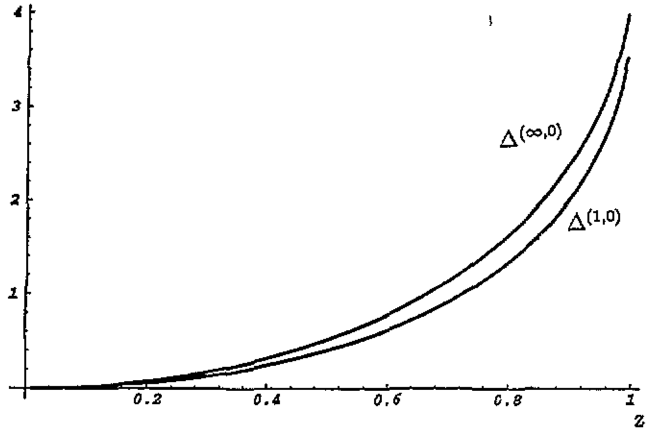

Figure 3. Plots of functions $\Delta^{(\infty,0)}(z)$ and $\Delta^{(1,0)}(z)$ in the interval $0 < z < 1$ for the spherical compactification.

(0.3 for $z = 0.5$ and 0.05 for $z = 0.25$) and the function $\Delta^{(\infty,0)}(z)$ *within a few per cent accuracy* represents *more* than just the first mode.

To obtain more illustrative characteristics, we introduce the quantities

$$\epsilon_N(z) \equiv \frac{\Delta^{(N,0)}(z)}{\Delta^{(\infty,0)}(z)} \tag{34}$$

which show the relative contributions of the first N heavy Kaluza–Klein modes. The plots for some $\epsilon_N(z)$ are presented in figure 4 for $0 < z < 1$. We conclude that with an accuracy of about $5 \div 10\%$, the function $\Delta^{(\infty,0)}(z)$ in the range $s \sim 0.8M_1^2 \div 2.4M_1^2$ actually shows the presence of at least $3 \div 4$ first heavy modes in the theory.

Another interesting question is how the contribution of the heavy Kaluza–Klein modes depends on the *topology* of the space of extra dimensions. Here, we restrict ourselves only to comparison of the results for $\Delta_{\mathrm{S}}^{(\infty,0)}(z)$ calculated in this article ('S' stands for 'spherical compactification'), with the behaviour of $\Delta_{\mathrm{T}}^{(\infty,0)}(z)$ for the toroidal compactification obtained

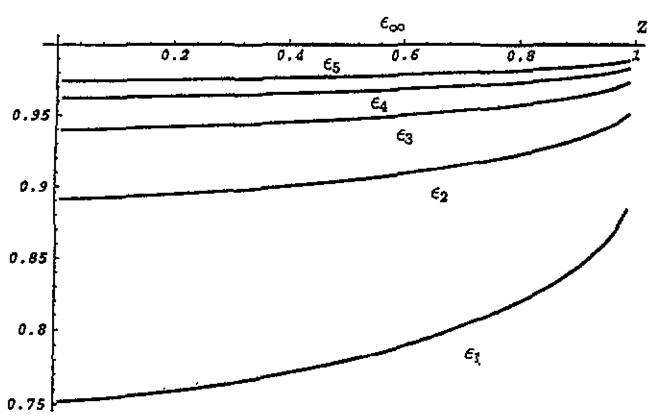

Figure 4. Plots of the functions $\epsilon_N(z)$ defined by equation (34) for $N = 1, 2, 3, 4, 5$; $\epsilon_\infty(z) \equiv 1$.

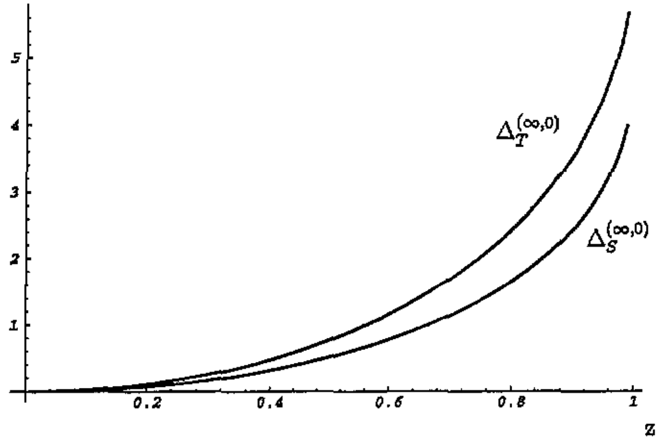

Figure 5. Plots of functions $\Delta_{\rm S}^{(\infty,0)}(z)$ and $\Delta_{\rm T}^{(\infty,0)}(z)$ in the interval $0 < z < 1$ for compactifications of extra dimensions to the two-dimensional sphere S^2 and to the two-dimensional torus T^2, respectively. In both cases the dimensionally reduced models are the same and scales characterizing these manifolds are chosen in such a way that $2M_{\rm S}^2 = M_{\rm T}^2$, so that the masses of the first heavy particles of the Kaluza–Klein towers are equal.

in [12] for a similar model but with a spacetime $M^4 \times T^2$, where T^2 is the two-dimensional torus. It is clear that the results of the comparison depend on the relation between the inverse radius of the sphere $M_{\rm S} = L^{-1}$ (for further discussion we have attached the index 'S' to it) and the scale $M_{\rm T}$ equal to the inverse radius of the circles forming the torus $T^2 = S^1 \times S^1$. For the present analysis, we assume that in both compactifications, the dimensionally reduced models, i.e. zero-mode sectors of the initial theories, coincide and that the masses of the first heavy modes are the same. These imply that, in both cases, the mass m_0 of the zero mode is the same and

$$2M_{\rm S}^2 = M_{\rm T}^2. \tag{35}$$

Plots of functions $\Delta_{\rm S}^{(\infty,0)}$ and $\Delta_{\rm T}^{(\infty,0)}$ for $0 < z < 1$ are presented in figure 5. The difference between the curves is quite noticeable (for example, $\Delta_{\rm T}^{(\infty,0)} - \Delta_{\rm S}^{(\infty,0)} = 0.25$ for $z = 0.5$ and it is equal to 0.06 for $z = 0.25$). This is due to the difference in the spectra of the Laplace operator on the sphere and on the torus. Namely, for the two-dimensional sphere S^2, the eigenvalues $\lambda_l(S^2)$ of the Laplace operator, the squares of the masses of the Kaluza–Klein modes determined by them and the multiplicities $d_l(S^2)$ of the eigenvalues are given by (cf equation (6))

$$(M_l^{(S)})^2 = m_0^2 + \lambda_l(S^2)M_{\rm S}^2 \qquad \lambda_l(S^2) = l(l+1) \qquad d_l(S^2) = 2l+1 \tag{36}$$

respectively. For the two-dimensional torus T^2, the analogous relations are:

$$(M_n^{(T)})^2 = m_0^2 + \lambda_n(T^2)M_{\rm T}^2 \qquad \lambda_n(T^2) = n_1^2 + n_2^2 \tag{37}$$

where $n = (n_1, n_2)$ is a two-vector labelling eigenvalues $-\infty < n_i < \infty$ $(i = 1, 2)$ and $d_{|n|}(T^2)$ is the number of such vectors with the same length $|n| = \sqrt{n_1^2 + n_2^2}$. Using

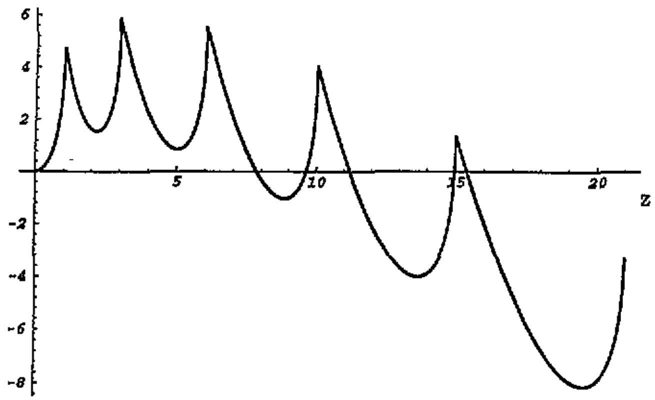

Figure 6. Plot of function $\Delta_{\rm S}^{(\infty,0)}(z)$ for the spherical compactification for $0 < z < 21$, i.e. up to the threshold of the sixth heavy mode.

expansions similar to (26) and equation (32), we get that, for $|z| < 1$,

$$\Delta_\alpha^{(\infty,0)}(z) \approx \frac{4}{9}\left(z\frac{M_{\rm T}^2}{M_\alpha^2}\right)^2 \zeta(2|K_\alpha). \tag{38}$$

Here, the index α labels the type of compactification, e.g. $\alpha = S$ for the case of the sphere and $\alpha = T$ for the case of the torus, and $\zeta(s|K_\alpha)$ is the zeta function of the Laplace operator on the manifold K_α [20] (see also [26])

$$\zeta(s|K_\alpha) = \sum_n{}' \frac{1}{(\lambda_n(K_\alpha))^s} \tag{39}$$

where the prime means that the term corresponding to the zero eigenvalue is absent from the sum. For $K_{\rm S} = S^2$, this function can be expressed in terms of the derivative of the generalized Epstein–Hurwitz zeta function (27):

$$\begin{aligned}\zeta(s|S^2) &= \sum_{l=1}^{\infty} \frac{d_l(S^2)}{[\lambda_l(S^2)]^s} = \sum_{l=1}^{\infty} \frac{2l+1}{[l(l+1)]^s} \\ &= -\frac{1}{s-1}\frac{\partial}{\partial a}F(s; a, -\tfrac{1}{4})|_{a=1/2}.\end{aligned} \tag{40}$$

Taking into account relation (35) between $M_{\rm S}^2$ and $M_{\rm T}^2$, we obtain an approximate expression connecting the ratio of the contributions of the two Kaluza–Klein towers of particles, corresponding to the spherical and toroidal compactifications, with the characteristics of the Laplace operator on these manifolds

$$\frac{\Delta_{\rm S}^{(\infty,0)}(z)}{\Delta_{\rm T}^{(\infty,0)}(z)} \approx \frac{4\zeta(2|S^2)}{\zeta(2|T^2)} \approx 0.66. \tag{41}$$

The results of our numerical computations (figure 5) are, with good accuracy, in accordance with formula (40).

Table 1. Table of values.

z	0.1	0.25	0.5	0.75	0.99
$\Delta_S^{(\infty,0)}$	0.0181	0.1165	0.5111	1.361	3.969
$\Delta_S^{(1,0)}$	0.0136	0.0888	0.3982	1.103	3.508
$\Delta_T^{(\infty,0)}$	0.0271	0.1749	0.7626	2.008	5.650

Using equation (45) and similar formulae for higher momentum intervals, we have also calculated the function $\Delta_S^{(\infty,0)}(z)$ for $1 < z < 21$. Its plot is presented in figure 6 and demonstrates the appearance of resonances. The peaks of the curve correspond very approximately (since $m_0 \neq 0$) to the values $s = 4M_l^2$ or $z = l(l+1)/2$ for $l = 1, 2, 3, 4, 5, 6$, i.e. to the thresholds of heavy-mode particle creation.

5. Conclusions

In this paper, we have studied the behaviour of the total cross section for the scattering of two light particles in an effective theory in four dimensions, obtained from the six-dimensional scalar theory with spherical compactification of two extra dimensions.

Even if our model cannot be termed directly as being physical, we do believe that the effect we have calculated is of a very general nature, and that it will also take place in more realistic theories. Provided that the scale L^{-1} is of the order of M_{SUSY} (see the introduction for a discussion of such a possibility) and that we calculate a real process, the corresponding results can, *in principle*, be used for comparison with actual experimental data. The idea is as follows.

We assume that the low-energy sector of the theory is already well determined. Therefore, the value of the renormalized coupling constant g is known and the total cross section $\sigma^{(0)}(s)$ can be calculated with sufficient accuracy. Experimentally, one should measure the total cross section $\sigma^{\mathrm{exp}}(s)$ and compute the quantity

$$\Delta^{\mathrm{exp}}(s) = 16\pi^2 \frac{\sigma^{\mathrm{exp}}(s) - \sigma^{(0)}(s)}{g\sigma^{(0)}(s)}$$

(cf (31)). If, above the threshold for the lightest particle, one has that $\Delta^{\mathrm{exp}}(s) = 0$, then there is no evidence of heavy Kaluza–Klein modes at the given energies.

On the contrary, $\Delta^{\mathrm{exp}}(s) \neq 0$ indicates the presence of heavier particles. The obvious next step will be to see which curve $\Delta_K^{(N,0)}(s)$ fits the experimental data best. If it is the curve with $N = \infty$ (or sufficiently large N) for a certain manifold K, this fact will be considered as indirect evidence of the *multidimensional* nature of the interactions—at least within the *framework of the given class of models and for that type of compactification.* As we have shown some information about the regularity of (at least the first) mass eigenvalues and their multiplicities is already encoded in the behaviour of the $\Delta^{(\infty,0)}$ function for $s \leqslant 4M_1^2$, up to a certain accuracy. Thus, one can hope that from this it will be possible to distinguish Kaluza–Klein-type models from, say, grand unified models with a finite number of heavy particles. To have more concrete criteria for such distinction, further investigation is needed. It would also be interesting to understand whether one can possibly distinguish between multidimensional models and certain models with an infinite tower of composite particles.

Having in mind the possibility of $L^{-1} \sim M_{\mathrm{SUSY}}$, we have choosen for our computations the values of the parameters which can mimic a physical situation with, for example,

$m_0 = 100$ GeV and $M = 10$ TeV, and with the charge g renormalized at the low-energy point $\sqrt{\mu_s^2} = 10$ GeV (see equations (33)). The results suggest that, indeed, the effect can be quite noticeable, even for energies below the threshold of the first heavy particle (see figure 1).

Our results also show that one can distinguish between different types of compactification of the extra dimensions. The main contribution, for energies below the threshold of the first heavy-mass state, is basically determined by the zeta function $\zeta(2|K)$, uniquely associated with the two-dimensional manifold K through the spectrum of the Laplace operator on it. This provides, by the way, a further example of the relevance of the concept of zeta functions in high-energy physics.

Of course, one should not forget that the effect studied here is a one-loop order effect and, obviously, therefore difficult to detect experimentally. Because of this, an interesting possibility, in a more realistic model, would be to consider specific processes for which the tree approximation is absent and the leading contribution is given by the one-loop diagrams even in the zero-mode sector of the theory. Such processes are clearly more sensitive to the heavy Kaluza–Klein modes.

A few remarks on the influence of the choice of the renormalization scheme and of the renormalization point follow. Conditions (9) and (10) define a renormalization scheme with manifest decoupling of heavy Kaluza–Klein modes. This means that in the limit sL^2, $\mu_s L$, $m_0 L \to 0$, the Green function and the cross section coincide with those in the ordinary $g\phi^4$ theory in four dimensions, renormalized according to the second equality in equation (9). In the case of a renormalization scheme without manifest decoupling, in order to have a physical interpretation of the low-energy limit of a given multidimensional model, further finite renormalizations of the coupling constants and masses will be necessary. Once they are performed, the results will be equivalent to a certain scheme with manifest decoupling. See the discussion of these issues in [10, 11].

Choosing a renormalization scheme, with manifest decoupling different from (9) and (10), would lead to finite renormalizations of the coupling constants g and λ_2 in the next-to-leading order, i.e. $\sim g^2$. The renormalization of λ_2 would not affect the result (32) due to the mass shell cancellations among the s-, t- and u-channels explained in section 2. The finite renormalization of g would amount to a finite addition (containing terms constant and linear in s) to the right-hand side of (32), which vanishes in the limit sL^2, $\mu_s L$, $m_0 L \to 0$.

Provided that the renormalization scheme is that given by equations (9) and (10), the result depends on the choice of renormalization point. The dependence of the renormalized coupling constants on μ_s is determined by renormalization-group equations similar to those for the torus compactification derived in [10, 11]. Changing μ_s again amounts to a finite renormalization of g and λ_2, and a corresponding finite addition to $\Delta^{(\infty,0)}$, as discussed above. Here, two final comments are in order. First, the value of $g(\mu_s)$ can always be chosen to be sufficiently small so that the Landau pole of the solution of the renormalization-group equations for the running coupling constant g, which is independent of λ_2 (see [11]), is located much further away than the thresholds of the first heavy modes. Thus, we expect that a finite addition to $\Delta^{(\infty,0)}$, though power-like rather than logarithmic in μ_s, will not considerably change our result around the threshold of the first heavy particle. Second, although function (32) does not depend on the renormalization of λ_2, in our one-loop calculations in section 2, we assumed that $\lambda_2 \leqslant g^2$. If $\lambda_2 = 0$, or if it is finely tuned to be small enough, the inequality for the running coupling constants remains valid at least until the threshold of the first heavy particle is reached. A more detailed analysis of the contributions to $\Delta^{(\infty,0)}$ due to the finite renormalizations, as well as of the effect of two-loop corrections, is beyond the scope of the present paper.

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Appendix

Some useful and mathematically elegant expressions for functions $F(s;a,b)$ (equation (27)) and $h^{(1)}(s;1/2,b)$ (equation (29)) have been obtained in [27] (see also [21]). In particular, use of the zeta-function regularization theorem yields [27, 21]

$$F(s;a,b) = \frac{b^{1-s}}{\Gamma(s-1)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+s-1)}{n!} b^{-n} \zeta(-2n,a) + \frac{\sqrt{\pi}}{2} b^{3/2-s} \frac{\Gamma(s-\frac{3}{2})}{\Gamma(s-1)} + \frac{2\pi^{s-1}}{\Gamma(s-1)} b^{3/4-s/2} \sum_{n=1}^{\infty} n^{s-3/2} \cos(2\pi na) K_{s-3/2}(2\pi n\sqrt{b})$$

which, in spite of the equality sign, should be understood as an asymptotic expression, *not* as a convergent series expansion. (It is noticeable that this very non-trivial result *cannot* be obtained by using just the Jacobi theta-function identity.) The optimal cut of this series has been numerically studied in detail [24]. Now taking into account equation (30), we obtain

$$h^{(1)}(s;1/2,b) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!} \frac{\Gamma(n+s-1)}{\Gamma(s)} b^{1-n-s} \tilde{\zeta}(-2n,1/2) \tag{42}$$

where

$$\tilde{\zeta}(s,a) \equiv \frac{\partial}{\partial a} \zeta(s,a).$$

$\zeta(s,a)$ is the Hurwitz (also called the Riemann-generalized) zeta function defined for $s > 1$ by the series

$$\zeta(s,a) \equiv \sum_{l=0}^{\infty} \frac{1}{(l+a)^s}.$$

Actually, the following simple relation holds:

$$\tilde{\zeta}(s,a) = -s\zeta(s+1,a) \tag{43}$$

and, for the few first terms in equation (42) (providing the best cut of the asymptotic series), we get

$$h^{(1)}(s;1/2,b) = b^{1-s}\left[\frac{1}{s-1} + 2b^{-1}\zeta(-1,1/2) - 2sb^{-2}\zeta(-3,1/2) + s(s+1)b^{-3}\zeta(-5,1/2) - \cdots\right]. \tag{44}$$

We have used the fact that the coefficient $-\tilde{\zeta}(0, 1/2) = 1$, however

$$\tilde{\zeta}(0, a) = \frac{\partial}{\partial a}\zeta(0, a) = \frac{\partial}{\partial a}(\tfrac{1}{2} - a) = -1.$$

The same result is obtained from equation (43) in the limit $s \to 0$. Numerical values of the Hurwitz zeta function are

$$\zeta(-1, 1/2) = \tfrac{1}{24} \qquad \zeta(-3, 1/2) = -\tfrac{7}{960} \qquad \zeta(-5, 1/2) = \tfrac{31}{8064}$$
$$\zeta(-7, 1/2) = -\tfrac{127}{30\,720} \qquad \zeta(-9, 1/2) = -\tfrac{511}{67\,584}, \ldots.$$

We see, in fact, that the best cut of asymptotic expansion (44) is obtained after $\zeta(-5, 1/2) = 0.003\,84$.

In equation (28), function $\zeta^{(1)}(s, a)$ is the truncated Hurwitz zeta function

$$\zeta^{(k)}(s, 1/2) \equiv \zeta(s, 1/2) - \sum_{n=0}^{k-1}(n + 1/2)^{-s} \qquad k = 1, 2, 3, \ldots$$

whereas the numerical values of the coefficients c_k are

$$c_1 = \frac{2}{3} \qquad c_2 = \frac{2^3}{3\cdot 5} \qquad c_3 = \frac{2^6}{3\cdot 5\cdot 7} \qquad c_4 = \frac{2^8}{5\cdot 7\cdot 9}$$
$$c_5 = \frac{2^{12}}{5\cdot 7\cdot 9\cdot 11} \qquad c_6 = \frac{2^{20}}{7\cdot 9\cdot 11\cdot 13}, \ldots.$$

For $4(m_0^2 + 2M^2) \leqslant p^2 < 4(m_0^2 + 6M^2)$ the following convergent expansion can be obtained for the functions (23) and (25):

$$\begin{aligned}
\Delta I\left(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon\right) &= \mathrm{i}\pi^{2-\epsilon}\Gamma(\epsilon)[2\zeta(-1, \tfrac{1}{2}) - 1] - 2\mathrm{i}\pi^2\sum_{l=1}^{\infty}(l + \tfrac{1}{2})\ln[(l + \tfrac{1}{2})^2 + b] \\
&\quad - \mathrm{i}\pi^2\Bigg[\mathrm{i}\pi\sqrt{1 - \frac{4M_1^2}{p^2}} + \sqrt{1 - \frac{4M_1^2}{p^2}}\ln\frac{p^2}{M_1^2} - \frac{2M_1^2}{p^2}\left(1 - \frac{M_1^2}{2p^2} + \cdots\right) \\
&\quad - 2 + 2\sum_{l=2}^{\infty}(l + \tfrac{1}{2})J_2\left(\frac{p^2}{4M_l^2}\right)\Bigg] + O(\epsilon) \\
&= 2\mathrm{i}\pi^2\Bigg[- \tfrac{11}{24}\pi^{-\epsilon}\Gamma(\epsilon) + 2\zeta'(-1, 1/2) + \ln 2 \\
&\quad + \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\zeta^{(1)}(2k - 1, 1/2)\left(\frac{m_0^2}{M^2} - \frac{1}{4}\right)^k - \frac{3}{2}\sqrt{1 - \frac{4M_1^2}{p^2}} \\
&\quad + \frac{1}{2}\sqrt{1 - \frac{4M_1^2}{p^2}}\ln\frac{p^2}{M_1^2} - \frac{M_1^2}{p^2}\left(1 - \frac{M_1^2}{2p^2} + \cdots\right) \\
&\quad + \sum_{k=1}^{\infty}3^k c_k\left(\frac{m_0^2}{6M^2} + 1\right)^k h^{(2)}\left(k; \frac{1}{2}, \frac{m_0^2}{M^2} - \frac{1}{4}\right)\left(\frac{p^2}{4M_2^2}\right)^k + O(\epsilon^2)\Bigg].
\end{aligned} \qquad (45)$$

Here

$$h^{(2)}(k;1/2,b)\equiv\frac{1}{1-k}\frac{\partial}{\partial a}\sum_{l=2}^{\infty}\frac{1}{[(l+a)^2+b]^k}\bigg|_{a=1/2}.$$

A useful asymptotic expression for the derivative $\zeta'(-1,1/2)$ can be found in [28]. By analogous methods, expansions for $p^2<0$ can be performed.

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